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Smooth curves possessing a small number of linear systems computing the gonality

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1. INTRODUCTION

Let C be a smooth irreducible complete curve of genus g defined over the field $\mathbf{C}$ of the complex numbers. It is well-known that the gonality of a general such curve is $[(g+3)/2]$. Let M_g be the coarse moduli space of curves of genus g and for $2 \leq k \leq [(g+1)/2]$ let $M_{g,k}$ be the locus of k-gonal curves. This locus is irreducible, its dimension is $2g + 2k - 5$. It is known that a general element of $M_{g,k}$ has a unique g^1_k (see [2]). In this paper we make some remarks on the number of linear systems g^1_k a k-gonal curve can have.

For an integer m we introduce $M_{g,k}(m) = \{C \in M_{g,k} : C$ has exactly m linear systems g^1_k; each one of them is of type I and they are mutually independent$\}$. A linear system g^1_k is said to be of type I if $\dim|2g^1_k| = 2$. This condition is equivalent to: g^1_k is not the limit of 2 different linear systems g^1_k in a family of curves (see [6]). So, linear systems g^1_k of type I are those that should be computed with multiplicity 1. Two linear systems g_1 and g_2 on C are called dependent if there is a non-trivial morphism $h : C \to C'$ and linear systems g'_1 and g'_2 on C' such that $g_i = h^*(g'_i)$.

At first thought, it is reasonable to expect that, if $M_{g,k}(m)$ is not empty for some $m > 1$ then also $M_{g,k}(m-1)$ is not empty and $M_{g,k}(m)$ belongs to its closure. However this is not always the case. It is very easy to prove that $M_{7,4}(3)$ is not empty while $M_{7,4}(2)$ is empty: 4-gonal curves of genus 7 with exactly 2 lin-

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ear systems g_4^1 are limits of 4-gonal curves of genus 7 with exactly 3 linear systems g_4^1 and one of the two linear systems g_4^1 is not of type I. The case $g = 7$; $k = 4$ is a case where a priori it would be reasonable to expect that $M_{g,k}(2)$ is not empty. In [10] we prove that in all other such cases $M_{g,k}(2)$ is indeed not empty: for all integers g and k satisfying $4 \leq k \leq [(g+1)/2]$ and $8 \leq g \leq (k-1)^2$ there exists a k-gonal curve of genus g possessing exactly 2 linear systems g_k^1 both of type I and independent, i.e. $M_{g,k}(2)$ is not empty.

In §2 of this paper we investigate 5-gonal curves of genus at least 10. As in the case $k = 4$, $g = 7$ we find gaps in the possible number of linear systems g_5^1 of type I: $M_{10,5}(5)$ is not empty but $M_{10,5}(4)$ and $M_{10,5}(3)$ are empty. Also $M_{11,5}(4)$ is not empty but $M_{11,5}(3)$ is empty.

In §3 we are able to study $M_{g,k}(3)$ for large values of g. We prove non-emptiness but we also prove that in many cases $M_{g,k}(3)$ is not irreducible. This is different from the situation if $m = 2$: in [4] Ballico and Keem prove that $M_{g,k}(2)$ is irreducible. For a few values of g; k and m we are able to study $M_{g,k}(m)$ with $m \geq 4$. Again we find situations that are not as one would expect. Nevertheless I expect that there is a general rule concerning the non-emptiness of $M_{g,k}(m)$ and the relations with respect to their closures. At the moment I am not able to make a conjecture about it.

For the general gonality $k = [(g+3)/2]$ the situation is as follows. If g is odd and C has gonality k then C has infinitely many linear systems g_k^1. If g is even and C has gonality k either C has infinitely many linear systems g_k^1 (examples of such curves are constructed in [13]) or C has at most $(2k-2)!/[(k-1)!k!]$ linear systems g_k^1. In general C has exactly $(2k-2)!/[(k-1)!k!]$ linear systems g_k^1. In that case each one of them is of type I. If C has less than $(2k-2)!/[(k-1)!k!]$ linear systems g_k^1 then at least one of them is not of type I.

In §4 we prove some results concerning curves with special plane models used in §3.

We say that a plane curve is nodal if the only singularities of the curve are ordinary nodes; it has simple singularities of multiplicity 2 if the only singularities are ordinary nodes and ordinary cusps. If Γ is a plane curve then we say that a point s on Γ is an ordinary singularity of multiplicity m if s is a point of multiplicity m on Γ and the tangent cone of Γ at s is a reduced union of m lines. The survey paper [11] can be considered as a more extended introduction to this paper.

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2. PENTAGONAL CURVES

Remarks 2.1. Let C be a smooth 5-gonal curve of genus $g \geq 8$. If C has infinitely many linear systems g_5^1 then $g = 10$ and C is isomorphic to a smooth plane curve of degree 6. This follows from results in [1] ($g \leq 10$; if $g = 10$ then C is isomorphic to a smooth plane curve of degree 6); [6] ($g \neq 9$); [16] and [5] ($g \neq 8$). In [7] it is proved that, if $g = 10$ and C has at least 6 linear systems g_k^1

then C has infinitely many linear systems g^1_5. On the other hand, if C is the normalization of a plane curve Γ of degree 7 with exactly 5 simple singularities of multiplicity 2, then C is a 5-gonal curve of genus 10 possessing exactly 5 linear systems g^1_5 (see [9] for the study of base point free linear systems on curves with plane models with simple singularities of multiplicity 2). Also each g^1_5 is of type I. (Fix a singular point s; the canonically adjoint curves of Γ containing 2 general lines through s are determined by conics through the other 4 singular points. Since those 4 singular points are not on one line they define a 1-dimensional family of conics. From the Riemann–Roch Theorem if follows that $\dim|2g^1_5| = 2$.) Conversely, in [7] it is proved that any smooth 5-gonal curve of genus 10 having exactly 5 linear systems g^1_5 is the normalization of such a plane curve Γ. This implies that $M_{10,5}(5)$ is not empty and it is irreducible. As mentioned in the introduction we also know that $M_{10,5}(2)$ and $M_{10,5}(1)$ are not empty. From Theorem 2.2 it follows that $M_{10,5}(3)$ and $M_{10,5}(4)$ are empty. Of course, one can specialize a plane nodal curve Γ of degree 7 with 5 nodes to a plane curve Γ_0 having exactly 3 (resp. 1) ordinary nodes and 1 (resp. 2) tacnodes. In this case, the normalization C possesses exactly 4 (resp. 3) linear systems g^1_5 but 1 (resp. 2) of them are not of type I (they are limit of 2 different linear systems g^1_5 in a family of curves). On M_{10} those curves belong to the closure of $M_{10,5}(5)$.

Theorem 2.2. *Let C be a smooth curve of genus 10 and assume C is not isomorphic to a smooth plane curve of degree 6 and it is not the normalization of a plane curve of degree 7. Then C has at most 2 base point free linear systems g^1_5.*

Proof. Suppose C has 3 base point free linear systems g^1_5 – say g_1; g_2 and g_3. From [1] one finds the existence of $\varepsilon \geq 0$ with $\dim(|g_1 + g_2|) = 3 + \varepsilon$; $\dim(|g_1 + g_2 + g_3|) \geq 6 + \varepsilon$, hence $\dim(|K_C - (g_1 + g_2)|) = 2 + \varepsilon$; $\dim(|K_C - (g_1 + g_2 + g_3)|) \geq \varepsilon$.

Take general points $P_1; \ldots ; P_\varepsilon$ on C and let $M = |K_C - (g_1 + g_2 + P_1 + \ldots + P_\varepsilon)|$. Hence $\dim(M) = 2$ and there exists an effective divisor D_3 such that $g_3 + D_3 \subset M$. Since g_3 is a base point free g^1_5 and 5 is a prime number, the linear system M is simple. Our assumptions on C imply $\deg(D_3 + g_3) = \deg(M) \geq 8$, hence $\deg(D_3) \geq 3$. But $\deg(K_C - (g_1 + g_2)) = 8$, hence $\varepsilon = 0$; $\dim(|g_1 + g_2|) = 3$, so $M = |K_C - (g_1 + g_2)|$ and $\dim(|g_1 + g_2 + g_3|) = 6$, hence $|K_C - (g_1 + g_2 + g_3)| = \{D_3\}$ and $\deg(D_3) = 3$.

Let $f : C \to C' \subset \mathbf{P}^2$ be the morphism associated to M. The divisor D_3 on C corresponds to a point p of multiplicity 3 on C' such that the pencil of lines through p induces the linear system g_3 on C. Suppose p' is another singular point on C' (possibly infinitely near to p; since $[(7.6)/2] - 3 = 18 > 10$, such point p' exists).

Suppose p' has multiplicity at least 4 (in particular, p' is not infinitely near to p). Then the pencil of lines through p' induces a linear system g^1_e on C with $e < 5$. From [1] one finds $\dim(|g_1 + g_2 + g_3 + g^1_e|) \geq 10$. It follows that $\dim(|K_C - (g_1 + g_2 + g_3 + g^1_e)|) = \dim(|K_C - (g_1 + g_2 + g_3)| = 0$, so each divisor of g^1_e would be contained in D_3, of course this is absurd. Hence the multiplicity of p' is at most 3. Let D' be the effective divisor on C coming from p'. Let L be the line on $\mathbf{P}^2$ through the points p and p'. Let $E = L \cdot C - D_3 - D'$, then $\deg(E) \in \{2; 3\}$ (by $L \cdot C$ we mean the pull-back of the divisor $L \cdot C'$ to C). Consider the linear system $|M + (L \cdot C - D_3 - D')|$. It contains the linear system $|M| + E$.

If p' is not infinitesimal near to p then the pencil of lines through p' induces a base point free linear system $g^1_e (e \geq 5)$ on C. Write $L \cdot C = D_3 + D_1 (D_1 \in g_3)$; $L \cdot C = D' + D_2 (D_2 \in g^1_e)$. Then $2L \cdot C - D_3 - D' = D_1 + D_2$, hence $g^1_e + g_3 \subset |M + (L \cdot C - D_3 - D')|$. Since $g^1_e + g_3$ has no fixed points, it follows that $|M + (L \cdot C - D_3 - D')| \neq |M| + E$.

If p' is infinitesimal near to p, then $D_3 = D' + \mu$ with μ an effective divisor on C with $\deg(\mu) \leq 1$. In this case $2L \cdot C - D_3 - D' = D_3 + D' + 2E = \mu + 2(D' + E) \in \mu + 2g_3$. It follows that $|M + E|$ has at most 1 fixed point. Since $\deg(E) \geq 2$, it follows that $|M + E| \neq |M| + E$.

In both cases, we found an effective divisor E such that $\deg(E) \in \{2; 3\}$ with $|g_3 - E| \neq \emptyset$ and $\dim(|M + E|) \geq 3$. It follows that $\dim(|K_C - (M + E)|) = \dim(|g_1 + g_2 - E|) \geq 2 - \varepsilon'$ with $\deg(E) = 2 + \varepsilon'$, hence $\varepsilon' \in \{0; 1\}$.

Consider a morphism $g : C \to C'' \subset \mathbf{P}^3$ associated to $|g_1 + g_2|$. There exists a smooth quadric Q in $\mathbf{P}^3$ with $C'' \subset Q$ (the pencils of lines on Q induce g_1 and g_2 on C). We conclude that there exists a line L in $\mathbf{P}^3$ with $L \cdot C \geq E$ (again $L \cdot C$ is induced from the scheme-theoretic intersection $L \cap C''$). If it is impossible to take $L \subset Q$, then $\varepsilon' = 1$, hence $\deg(E) = 3$. If $\varepsilon' = 1$ but if it is possible to take L on Q, then we can find $\tilde{D}$ on g_1 or g_2 (say g_1) with $\tilde{D} \geq E$. In this case $|\tilde{D} + D'|$ contains $g_1 + D'$ and $|\tilde{D} - E| + g_3$. Since $\deg(\tilde{D} + D') = 7$, we obtain a simple base point free linear system g^2_6 or g^2_7 on C. This is a contradiction to the assumptions on C. If it is impossible to take $L \subset Q$ with $\varepsilon' = 1$, then $\deg(L \cdot C) \geq 3 > \deg(L \cdot Q) = 2$. This implies that some point of $L \cap C''$ is a singular point of C''. In case $\varepsilon' = 0$, then of course E also defines a singular point of C''. Hence we proved the existence of an effective divisor E' of degree 2 on C and $D \in g_3$ with $D = E' + E''$ (E'' an effective divisor of degree 3 on C) and $\dim(|g_1 + g_2 - E'|) = 2$.

Write $E'' = E''' + \tilde{P}$ with E''' an effective divisor of degree 2 on C. We find $\bar{D} \in |g_1 + g_2 - E'|$ with $\bar{D} = E''' + D'''$ and D''' an effective divisor on C. Consider the linear system $|g_1 + g_2 + \tilde{P}|$. Since $D''' + E''' + E' + \tilde{P} \in |g_1 + g_2 + \tilde{P}|$, we find $D''' + g_3 \subset |g_1 + g_2 + \tilde{P}|$. If $\tilde{P}$ would not be a fixed point of $|g_1 + g_2 + \tilde{P}|$ then $\tilde{P}$ is a fixed point of $|M|$. Because of the assumptions that is not possible. Since $\tilde{P}$ is a fixed point of $|g_1 + g_2 + \tilde{P}|$ we find $\tilde{P} \in D'''$ and we find an effective divisor F of degree 5 on C with $F + g_3 \subset |g_1 + g_2|$. Then there exists a line L' in $\mathbf{P}^3$ with $L' \cdot C = F$ and the pencil of planes in $\mathbf{P}^3$ through L' induces g_3 on C. Since $g_3 \notin \{g_1; g_2\}$, one has $L' \not\subset Q$. From this fact we are going to conclude that C'' has a point of multiplicity at least 3. Projection with this point as the center defines a linear system $g^2_e (e \leq 7)$ on C. Moreover for some effective divisors $F_1; F_2$ on C one has $g_i + F_i \subset g^2_e$, hence g^2_e is simple. This is a contradiction to our assumptions on C and the theorem is proved. So, we have to prove that $L' \cap C''$ contains a point of multiplicity at least 3 on C''.

If $L' \cap Q$ consists of 2 points then L' is not tangent to Q. The tangent cone to C'' at points of $L' \cap C''$ is a union of lines tangent to Q, hence they do not contain L'. This implies that $\deg(L' \cdot C)$ is the sum of the multiplicities on C'' of points in $L' \cap C''$.

Next suppose $L' \cap Q$ is one point s; hence L' is tangent to Q at $s \in C''$. Suppose the multiplicity of s on C'' is at most 2, so s defines a divisor P (C'' smooth at s); $2P$ or $P + Q$ with $P \neq Q$ (C'' is singular at s) on C. In the first two cases $F = 5P$; in the third case $F = xP + yQ$ with $x + y = 5$. In this last case we can assume that $x \geq 3$. The line L' is completely determined as the intersection of the planes in $\mathbf{P}^3$ corresponding to $|g_1 + g_2 - 2P|$ (resp. $|g_1 + g_2 - 3P|$; $|g_1 + g_2 - 2P|$). Since L' is not one of the lines in the rulings it follows that P is not a ramification point of g_1 or g_2 (resp. P is an ordinary ramification point of g_1 and g_2; P is not a ramification point of g_1 or g_2). Let D_0 be the divisor on C defined by the tangent plane to Q at s. It is the sum of the divisor in g_1 and in g_2 containing P. Since it is defined by a plane containing L', it has to contain $5P$ (resp. $5P$; $xP + yQ$). But we obtain that D_0 has multiplicity 2 (resp. 4; 2) at P. This gives a contradiction.

Example 2.3. (a) In [1] it is proved that a 5-gonal curve of genus 11 has at most 4 linear systems g^1_5. In [8] it is proved that a 5-gonal curve of genus 11 possessing 4 linear systems g^1_5 is birationally equivalent to a plane curve Γ of degree 7 with 4 simple singularities of multiplicity 2. The reader can easily check that on the normalization of such a curve Γ each linear system g^1_5 is of type I. It follows that $M_{11,5}(4)$ is not empty.

Now suppose C is a 5-gonal curve of genus 11 and $g_1; g_2$ and g_3 are 3 different linear systems g^1_5 on C. From [1] one finds that $\dim(|g_1 + g_2 + g_3|) \geq 6$ hence $\dim(|K_C - (g_1 + g_2 + g_3)|) \geq 1$. But $\deg(K_C - (g_1 + g_2 + g_3)) = 5$, so one finds another g^1_5, call it $g_4 = |K_C - (g_1 + g_2 + g_3)|$. In case $g_4 \in \{g_1; g_2; g_3\}$, say $g_4 = g_1$, then $|K_C - 2g_1| = |g_2 + g_3|$ and so $\dim(|K_C - 2g_1|) \geq 3$. It follows that $\dim(|2g_1|) \geq 3$, so g_1 is not of type I. It follows that $M_{11,5}(3)$ is empty.

Remember that we already know that $M_{11,5}(1)$ and $M_{11,5}(2)$ are not empty.

(b) In [1] it is proved that a 5-gonal curve of genus 12 has at most 3 linear systems g^1_5. In [8] it is proved that a 5-gonal curve of genus 12 possessing 3 linear systems g^1_5 is birationally equivalent to a plane curve Γ of degree 7 with 3 simple singularities of multiplicity 2, or to a plane curve Γ of degree 8 with 3 simple singularities of multiplicity 3. However a quadratic transformation interchanges both models. This implies that $M_{12,5}(3)$ is irreducible.

(c) Finally 5-gonal curves of genus $g \geq 13$ have at most 2 linear systems g^1_5; 5-gonal curves of genus $g \geq 17$ have a unique g^1_5.

Problem 2.4. Let $g = 9$. For what values m is $M_{9,5}(m)$ not empty? For 5-gonal curves this is the only genus that remains unsettled.

In [1] the following result is proved.

Proposition 3.1. *Let C be a smooth k-gonal curve of genus g and assume C possesses s mutually independent linear systems g^1_k. Write $k = m(s-1)+q$ for some integers q; m satisfying $-s+3 \leq q \leq 1$. Then $g \leq [m^2(s^2-s)+(2ms+q-2)(q-1)]/2 := g(s;k)$.*

Using plane models, Accola also shows that his inequality is sharp. In [8] we classify all such curves for which we have equality. We proved

Proposition 3.2. *Let $k; g; s; m; q$ be as in Proposition 3.1 with $s \geq 3$. Suppose C is a k-gonal curve of genus $g(s;k)$ having s mutually independent linear systems g^1_k. Then one of the following possibilities occur:*

(i) *C is birationally equivalent to a plane curve Γ of degree $k+m$ having s singular points of multiplicity m.*

(ii) *$q = 1$ and C is birationally equivalent to a plane curve of degree $k+m+1$ having s singular points of multiplicity $m+1$.*

(iii) *$s = 5$; k is even and C is birationally equivalent to a plane curve of degree $3k/2$ having 4 singular points of multiplicity $k/2$.*

Possibilities (i) and (ii) are the examples in Accola's paper. Also for $s = 5$ and k even and for $q = 1$ we obtain (at least) 2 components in the moduli space $M_{g(s;k)}$ for the locus of k-gonal curves possessing s mutually independent g^1_k.

The starting point for both proofs is a method of Castelnuovo. One considers complete linear systems obtained from integral combinations of the linear systems g^1_k. One finds lower bounds on their dimension and then one uses the residual linear system. Using Lemma 3.5, we are able to proceed making plane models also for some cases where $g < g(s;k)$. In this way we find new examples of 'gaps' in the possibilities for the number of linear systems g^1_k (of type I). The lemma is inspired by and can be deduced from the more general lemma in [14], Lemma 2.5, however we include a direct proof.

Definition 3.3. A set of different points $P_1; \ldots; P_x$ in $\mathbf{P}^N$ satisfies $CB(l)$ if for each $s \in H^0(\mathbf{P}^N; O_{\mathbf{P}^N}(l))$, if $\{P_1; \ldots; P_x\} \cap Z(s)$ contains at least $x-1$ points, then $\{P_1; \ldots; P_x\} \subset Z(s)$. (Here $Z(s)$ is the hypersurface of degree l in $\mathbf{P}^N$ defined by s.)

Remark 3.4. If $P_1; \ldots; P_x$ satisfies $CB(l)$ then $x \geq l+2$. This can be seen using hyperplanes in $\mathbf{P}^N$.

Lemma 3.5. *$N \geq 3$ and $l \geq 2$. If $x \leq 2l+3$ and $P_1; \ldots; P_x$ satisfy $CB(l)$, then those points are contained in a plane.*

We first prove

Sublemma 3.6. *$N \leq 2$. If $x \leq 2l+1$ and $P_1; \ldots; P_x$ satisfy $CB(l)$ then those points are on a line.*

Proof. For $l = 1$ this is trivial. Assume the property holds for $l-1$ instead of l. Let y be the maximal number of points $P_1; \ldots; P_x$ on a line and assume $y < x$. Assume $P_1; \ldots; P_y$ are on a line. Of course $y \geq 2$, hence $x - y \leq 2l + 1 - 2 = 2l - 1 = 2(l-1) + 1$. But the other $x - y$ points satisfy $CB(l-1)$, hence $x - y \geq l + 1$, i.e. $y \leq x - l - 1$ and also those points are on a line. It follows that $x - y \leq y$, hence $x \leq 2y$ and therefore $x \leq 2(x - l - 1)$ hence $x \geq 2l + 2$, a contradiction.

Proof of Lemma 3.5. If 4 points satisfy $CB(1)$ then those 4 points are on a plane.

Assume $x \leq 7$ points satisfy $CB(2)$. Let y be the maximal number of points $P_1; \ldots; P_x$ on a plane and assume $y < x$. Assume $P_1; \ldots; P_y$ are on a plane. Of course $y \geq 3$, hence $x - y \leq 4$. Those $x - y$ points satisfy $CB(1)$, hence $x - y \geq 3$ (Remark 3.4). If $x - y = 3$ then those points P_x; P_{x-1}; P_{x-2} are on a line (Sublemma 3.6). This proves $y \geq 4$, hence $x = 7$; $y = 4$. Let $P \in \{P_1; P_2; P_3; P_4\}$. Then P; P_x; P_{x-1}; P_{x-2} are on a plane. This plane does not contain any of the points $\{P_1; P_2; P_3; P_4\} \backslash \{P\}$ since $y = 4$. But then again Remark 3.4 implies that the points $\{P_1; P_2; P_3; P_4\} \backslash \{P\}$ are on one line, hence $P_1; P_2; P_3; P_4$ are on line, a contradiction to $y = 4$. Hence $x - y = 4$ and those points are on a plane because they satisfy $CB(1)$. It follows that $x - y = 4 \leq y$ hence $x \geq 8$, a contradiction.

Now take $l \geq 3$ and assume the lemma holds for $l - 1$ instead of l. Let z be the maximal number of points on a line. Suppose $P_1; \ldots; P_z$ are on a line. So $z \geq 2$ hence $x - z \leq 2l + 3 - z \leq 2l + 1 = 2(l-1) + 3$. Since the remaining $x - z$ points satisfy $CB(l-1)$, they are on a plane. But then the points not contained in that plane also satisfy $CB(l-1)$. Therefore if not all points are contained in that plane, then it follows that $z \geq l + 1$. This implies $x - z \leq 2l + 3 - l - 1 = l + 2 \leq 2(l-1) + 1$ since $l \geq 3$. Hence the remaining $x - z$ points are on a line. This proves that the x points are on the union of 2 lines. Since $CB(l)$ holds, both lines have to contain at least $l + 2$ points (otherwise one finds a contradiction using hyperplanes through one of those lines). Since $x \leq 2l + 3$ this is impossible, hence we obtain a contradiction.

Now we are ready to state the proposition that enables us to study k-gonal curves with exactly s linear systems g^1_k in some cases where $g < g(s; k)$.

Proposition 3.7. *Let C be a k-gonal curve and let $g_1; g_2$ be two independent linear systems g^1_k. Let $l = [(k-2)/2]$ and assume $g > 2l(k-1) - l^2$. If g_3 is another g^1_k on C then $|g_1 + g_2 - g_3|$ is not empty.*

Proof. Assume C is canonically embedded in $\mathbf{P}^{g-1}$. For each divisor $E \in g_3$ the geometric Riemann–Roch Theorem (see e.g. [3], p. 12) implies that the linear span $\langle E\rangle$ (intersection of all hyperplanes in $\mathbf{P}^{g-1}$ containing E as a closed subscheme) has dimension $k-2$. Since the gonality of C is k, for each subdivisor E' of E of degree $k-1$, the linear span $\langle E'\rangle$ has dimension $k-2$ too, hence $\langle E\rangle = \langle E'\rangle$.

Let $\phi : C \to Q \subset P^3$ be the morphism associated to $\langle g_1+g_2\rangle \subset |g_1+g_2|$ (here Q is a smooth quadric; the two rulings induce g_1 and g_2). From $l \leq k-1$ one finds $\dim|l(g_1+g_2)| \geq 2l+l^2$ (see [1], Lemma 4.2). The condition $g > 2l(k-1)-l^2$ implies that $|l(g_1+g_2)|$ is special. Let E be a general element of g^1_k and assume the set of k points $\phi(E)$ in $\mathbf{P}^3$ does not satisfy $CB(l)$. Then there exists a subdivisor $E' \subset E$ of degree $k-1$ and $D \in |l(g_1+g_2)|$ such that $D \cap E = E'$. In the canonical embedding this implies $\langle E'\rangle \subset \langle D\rangle$, hence $\langle E\rangle \subset \langle D\rangle$. Let $Q = E - E'$. Since $Q \notin D - E$ it follows that $\langle D+Q\rangle = \langle D\rangle$. Then the geometric Riemann–Roch Theorem implies $\dim(|D+Q|) = \dim(|D|)+1$ and from the Riemann–Roch Theorem we obtain $\dim(|K_C - l(g_1+g_2)-Q|) = \dim(|K_C - l(g_1+g_2)|)$. Hence Q is a fixed point of $|K_C - l(g_1+g_2)|$. But $l(g_1+g_2)$ is special, hence $|K_C - l(g_1+g_2)|$ has only finitely many fixed points. Since Q belongs to a general element E of g_3, we obtain a contradiction. Hence the points of $\phi(E)$ are k points satisfying $CB(l)$. The inequality $l \geq (k-3)/2$ imply that we can use Lemma 3.5: those k points are on a plane. This implies the proposition.

Remark 3.8. In case $\dim(|g_1+g_2|) \geq 4$ then Castelnuovo's bound (see [3]), implies $g \leq [3m'(m'-1)]/2 + m'\varepsilon'$ with $2k-1 = 3m'+\varepsilon'$ with $0 \leq \varepsilon' < 3$. This inequality is contradicted by the conditions in Proposition 3.7 except for the cases $k=9$; $g=40$ and $k=7$; $g=21$ and 22. However in those cases we can exclude the possibility $\dim(|g_1+g_2|) \geq 4$ because of the following argument (due to the referee). If $\phi : C \to \mathbf{P}^4$ is a morphism associated to a base point free 4-dimensional linear subsystem of $|g_1+g_2|$ then $\phi(C)$ is contained in a cubic surface in $\mathbf{P}^4$. This follows from a Theorem of Eisenbud and Harris in [12]; see also [3], Chapter III, Theorem 2.7. The example in [15] implies that the gonality of C is at most $2(2k-1)/4 = k-1/2 < k$. This is a contradiction.

Construction 3.9. We assume the inequalities of Proposition 3.7 hold. Let $\phi : C \to Q \subset \mathbf{P}^3$ be as in the proof of Proposition 3.7. Take $E \in g_3$ general and $D \in |g_1+g_2|$ such that $D \geq E$. Let $E' = D - E$. Then $|g_1+g_2-E'| = g_3$, hence E' imposes only 2 conditions on $|g_1+g_2|$. Hence there is a line $L' \subset \mathbf{P}^3$ such that for each plane $H \subset \mathbf{P}^3$ containing L' one has $\phi^*(H) \geq E'$. Since $g_3 \notin \{g_1; g_2\}$ this line cannot be contained in Q. Hence L' meets Q at 1 or at 2 points. Suppose L' meets Q at 1 point – say s_1. For $\phi(C)$ this point has multiplicity μ. Using s_1 for projecting to $\mathbf{P}^2$ we find a morphism $\phi' : C \to \mathbf{P}^2$ and L' gives a point on $\phi'(C)$ of multiplicity $k-\mu$. On Q this is a singular point for $\phi(C)$ in the 1° infinitesimal neighbourhood of s_1. It follows that $k-\mu \leq \mu$, hence $\mu \geq k/2$. Also the lines in the rulings on Q give rise to points of multiplicity $k-\mu$ on $\phi'(C)$. In case L' meets Q at 2 points – say s_1 and s_2 then at one of them

– say at s_1 – $\phi(C)$ has multiplicity $\mu \geq k/2$; at the other one $\phi(C)$ has multiplicity $k - \mu$.

Assume $\phi(C)$ has a point s of multiplicity $\mu > k/2$. If there would be another (possibly infinitesimal near) point s' of multiplicity $\mu' \geq k/2$ then the pencil of planes through the line ss' induces a linear system g^1_n for some $n < k$, a contradiction to C being k-gonal. Therefore for all linear systems g^1_k different from g_1 and g_2 there is a point s' of multiplicity $k - \mu$ (possibly infinitely near to s) such that the pencil of planes through ss' induces that linear system g^1_k. Projection on $\mathbf{P}^2$ with center s gives rise to a plane model Γ of degree $2k - \mu$. In case $\mu = k - 1$ then C becomes isomorphic to a smooth plane curve of degree $k + 1$ and in this case C has infinitely many linear systems computing the gonality. Taking into account the inequalities of Proposition 3.7 this is only possible for $k = 7$; $g = 21$. From now on we exclude that C is a smooth plane curve of degree 8. The number of linear systems g^1_k on C is equal to the number of singular points of multiplicity $k - \mu$ on Γ.

Assume $\phi(C)$ has points of multiplicity $\mu = k/2$. Then there is no point on $\phi(C)$ of multiplicity more than $k/2$, hence for each g^1_k on C there exist 2 singular points s_1 and s_2 of multiplicity $k/2$ on $\phi(C)$ such that the pencil of planes through s_1s_2 induces g^1_k. Projection with center s gives rise to a plane model Γ of degree $3k/2$. The lines in the ruling through s give rise to 2 singular points s_1 and s_2 of multiplicity $k/2$ on Γ. Each g^1_k on C is induced by the lines through a point of multiplicity $k/2$ on Γ or by a pencil of conics through $s_1; s_2$ and two other points of multiplicity $k/2$. This situation only occurs if C has exactly 3 or 5 linear systems g^1_k. (In case Γ has exactly 3 points of multiplicity $k/2$ then C has exactly 3 linear systems g^1_k; if Γ has exactly 4 points of multiplicity $k/2$ then C has exactly 5 linear systems g^1_k. If Γ would have at least 5 points of multiplicity $k/2$, say $s_1; s_2; \ldots; s_5$ then the pencil of conics through $s_2; s_3; \ldots; s_5$ would induce a linear system g^1_k that does not satisfy the description of the linear systems g^1_k on C. Hence Γ has at most 4 points of multiplicity $k/2$.)

Example 3.10. The previous construction can be used to describe plane models for k-gonal curves having exactly 3 linear systems g^1_k in case $g > 3k'^2 - 4k' + 1$ if $k = 2k'$ and $g > 3k'^2 - 2k' - 1$ in case $k = 2k' + 1$. In order to understand those inequalities you have to compare it with $g(3; k)$ (see Proposition 3.1). One obtains $g(3; 2k') = 3k'^2 - 3k' + 1$; $g(3; 2k' + 1) = 3k'^2$. In the following table, for some small gonalities, you find the range on g where we can apply Construction 3.9:

k	g
6	$17 \leq g \leq 19$
7	$21 \leq g \leq 27$
8	$34 \leq g \leq 37$
9	$40 \leq g \leq 48$
10	$57 \leq g \leq 61$
11	$65 \leq g \leq 75$
12	$86 \leq g \leq 91$

For those values of g we obtain that C is birationally equivalent to a plane curve Γ of some degree $2k-\mu$ for some $k>\mu\geq k/2$ having 3 singular points of multiplicity $k-\mu$. (We exclude the possibility that C is isomorphic to a smooth plane curve of degree 8.) We obtain a genus bound for such curves:

$$g\leq\frac{(2k-\mu-1)(2k-\mu-2)}{2}-3\frac{(k-\mu)(k-\mu-1)}{2}.$$

For the possibilities in the above table we obtain the following possibilities for μ:

k	μ	g
6	3	$17\leq g\leq 19$
6	4	18
7	4	$21\leq g\leq 27$
7	5	25
8	4	$34\leq g\leq 37$
8	5	$34\leq g\leq 36$
9	5	$40\leq g\leq 48$
9	6	$40\leq g\leq 46$
9	7	42
10	5	$57\leq g\leq 61$
10	6	$57\leq g\leq 60$
10	7	57
11	6	$65\leq g\leq 75$
11	7	$65\leq g\leq 73$
11	8	$65\leq g\leq 69$
12	6	$86\leq g\leq 91$
12	7	$86\leq g\leq 90$
12	8	$86\leq g\leq 87$

From the table we see that for some values of g within the range we are considering different values for μ are possible. In §4 we prove that such plane curves with exactly 3 singularities of multiplicity $k-\mu$ occur. From Proposition 3.7 it follows that their normalisations have exactly 3 linear systems g^1_k (for $k=6$; $\mu=4$ and $k=7$; $\mu=5$ we have to exclude $g<17$ and $g<25$ for this reason). From considerations in §4 it follows that it is possible to obtain that each g^1_k is of type I. Also from §4 it follows that a general curve belonging to a value for μ is not a specialization of curves belonging to another value of μ. In this way, one finds values g and k with g rather large, such that the locus $M_{g,k}(3)$ is not irreducible.

For $s\geq 4$, the inequalities from Proposition 3.1 imply that Proposition 3.7 can only be used in the following cases: $s=4$ and $(k;g)\in\{(6;17);\ (7;21);\ (7;22);\ (7;23);\ (7;24);\ (9;40);\ (9;41);\ (9;42);\ (9;43);\ (11;65);\ (11;66);\ (11;67);\ (13;96)\}$ and if $s\geq 5$ then $k=7$ and $g\leq 23$ or $k=9$ and $g=40$.
We examine some of those cases separately.

Example 3.11. 6-gonal curves of genus 17.

If C has exactly 3 (resp. 4) linear systems g_6^1, each one of type I, then C is birationally equivalent to a plane curve Γ of degree 9 having exactly 3 points of multiplicity 3 (resp. birationally equivalent to a plane curve of degree 8 having exactly 4 points of multiplicity 2). From the results in §4 it follows that both cases occur but $M_{17,6}(4)$ does not belong to the closure of $M_{17,6}(3)$.

Example 3.12. 7-gonal curves of genus $21 \leq g \leq 24$.

The normalization of a plane curve of degree 10 with 4 (resp. 5) points of multiplicity 3 has genus $g \leq (9.8/2) - 4.3 = 36 - 12 = 24$ (resp. $g \leq 21$). From results in §4 it follows that there exist 7-gonal curves of genus $21 \leq g \leq 24$ having exactly 4 linear systems g_7^1 each one of type I and having such a plane model. For $g = 24$ there also exist 7-gonal curves having exactly 4 linear systems g_7^1 each one of type I that are normalizations of plane nodal curves of degree 9 with 4 nodes. Those are not specializations of 7-gonal curves of genus 24 with 3 linear systems g_7^1. In particular $M_{24,7}(4)$ is not irreducible. A 7-gonal curve of genus 23 (resp. 22; 21) with exactly 5 (6; 7) linear systems g_7^1 is the normalization of a plane curve of degree 9 with simple singularities of multiplicity 2. A general such curve is not the specialization of 7-gonal curves having less than 5 (6; 7) but more than 2 linear systems g_7^1. There are no 7-gonal curves of genus 22 (resp. 21) with exactly 5 (6) linear systems g_7^1 each one of type I. From §4 it also follows that $M_{21,7}(5)$ is not empty.

Example 3.13. 9-gonal curves of genus $40 \leq g \leq 43$.

The normalization of a plane curve of degree 13 with 4 points of multiplicity 4 has genus $g \leq (12.11/2) - 4.6 = 42$. Those plane curves give rise to smooth 9-gonal curves of genus $g \in \{40; 41; 42\}$ having exactly 4 linear systems g_9^1, each one of them of type I. The normalization of a plane curve of degree 12 with 4 points of multiplicity 3 has genus $g \leq (11.10/2) - 4.3 = 43$. Those plane curves give rise to smooth 9-gonal curves of genus $g \in \{40; 41; 42; 43\}$ having exactly 4 linear systems g_9^1, each one of type I. The normalization of a plane curve of degree 12 with 5 points of multiplicity 3 has genus $g \leq 40$. Those plane curves give rise to smooth 9-gonal curves of genus 40 having exactly 5 linear systems g_9^1, each one of type I. The normalization of a general nodal curve of degree 11 with exactly 4 (resp. 5) nodes has genus $g = 41$ (resp. $g = 40$) having exactly 4 (resp. 5) linear systems g_9^1, each one of type I.

We obtain that $M_{42,9}(4)$, $M_{41,9}(4)$ and $M_{40,9}(4)$ are not irreducible. Also $M_{40,9}(5)$ is not irreducible. At least one of the components of $M_{41,9}(4)$ (resp. $M_{40,9}(5)$) (the component coming from the plane nodal models) is not contained in the closure of $M_{41,9}(3)$ (resp. $M_{40,9}(3)$).

Example 3.14. 11-gonal curves of genus $65 \leq g \leq 67$.

An 11-gonal curve of genus $65 \leq g \leq 67$ and having 4 linear systems g_{11}^1 each one of type I is the normalization of a plane curve of degree 15 with 4 singular points of multiplicity 4 or ($g = 65; 66$) the normalization of a plane curve of degree 14 with 4 singular points of multiplicity 3 or ($g = 65$) the normalization of a plane curve of degree 16 with 4 singular points of multiplicity 5. From §4 it follows that all possibilities occur. Also, it follows that for $g = 65; 66$ the locus of 11-gonal curves having 4 linear systems g^1_{11} – each one of type I – is not irreducible.

Remark 3.15. Suggestion of the referee.

The referee suggested improvements of the results in this paper using the Lopez–Pirola lemma (Lemma 2.5 in [14]) instead of Lemma 3.5 as follows.

Let C be a k-gonal curve and let $g_1; g_2$ be two independent linear systems g^1_k on C. Assume $k \geq 6; l = [k/3] + 1$ and $g > 2l(k-1) - l^2$. Let g_3 be another linear system g^1_k on C and take $E \in g^1_k$ general. Let ϕ be as in the proof of Proposition 3.7, then the Lopez–Pirola lemma implies that $\phi(E)$ are k points on a curve of degree 2. If $\phi(E)$ are on a union of 2 lines then at least one of those lines are on Q. Hence E would have more than one point in common with a divisor from g_1 or g_2 (say g_1). Then g_1 and g_3 are composed with a covering $C \to C$' of degree $k/2$ with C' an elliptic curve. (Indeed both g_1 and g_3 should be the pull-back of a linear system g^1_2 on C'.) In particular C has infinitely many linear systems g^1_k.

Assume that $\{g_1; g_2; g_3\}$ are mutually independent. Then $\phi(E)$ is on a conic, hence in a plane and we find $|g_1 + g_2 - g_3| \neq \emptyset$. Assume $\dim(|g_1 + g_2|) = r \geq 4$ and now, let $\phi : C \to \mathbf{P}^r$ be the associated morphism. The genus bound implies $g > (1/2)k^2 - k + 1$. First of all, the Castelnuovo bound for curves in a projective space implies $r = 4$. A theorem of Eisenbud and Harris in [12] (see also [3], Chapter III, Theorem 2.7) implies that $\phi(C)$ is on a surface of degree 3 in $\mathbf{P}^4$. The example in [15] implies that the gonality of C is less than k, a contradiction. Therefore $\dim(|g_1 + g_2|) = 3$ and we can use Construction 3.9 using the bound $g > 2l(k-1) - l^2$ with $l = [k/3] + 1$. Under the assumption that we are only considering mutually independent linear systems g^1_k this gives an improvement for Example 3.10 in case $k \geq 14$. Also it implies descriptions by means of plane models in case of s linear systems g^1_k with $3 < s < 10$. Indeed, let $k = m(s-1) + q$ with $-s + 3 \leq q \leq 1$. Then Proposition 3.1 induces an upper bound for g of type $g \leq ((s^2 - s)/2)m^2 +$ (lower order terms in m). The improved genus bound w.r.t. Proposition 3.7 becomes $g > (5(s-1)^2/9)m^2 +$ (lower order terms in m). In case $3 < s < 10$ one has $5(s-1)^2/9 < (s^2 - s)/2$.

4. STUDY OF PLANE CURVES

Proposition 4.1. *Let $k \geq 6$ and $\mu \geq k/2$; $\mu \leq k - 2$. For each $6k - 19 < g \leq [(2k - \mu - 1)(2k - \mu - 2) - 3(k - \mu)(k - \mu - 1)]/2$ there exists a plane curve Γ of degree $2k - \mu$ having 3 ordinary singular points of multiplicity $k - \mu$ such that the other singular points are ordinary nodes; the normalisation C has genus g and the pencil of lines through the singular points of multiplicity $k - \mu$ induce linear systems g^1_k on C of type I.*

Proof. Choose 3 points $s_1; s_2; s_3$ on $\mathbf{P}^2$ not on one line. Choose $L_1; \dots; L_{k-\mu}$ (resp. $L_{k-\mu+1}; \dots; L_{2(k-\mu)}; L_{2(k-\mu)+1}; \dots; L_{3(k-\mu)}$) lines through s_1 (resp. $s_2; s_3$). Let $L_{3(k-\mu)+1}; \dots; L_{2k-\mu}$ be $2\mu - k$ general lines in $\mathbf{P}^2$. The union of those lines is a plane curve Γ' of degree $2k - \mu$ having ordinary singular points of multiplicity $k - \mu$ at $s_1; s_2$ and s_3 and $3(k-\mu)^2 + 3(k-\mu)(2\mu - k) + (2\mu - k - 1)(2\mu - k)/2$ ordinary nodes as its other singularities.

Let $\pi : X \to \mathbf{P}^2$ be the blowing-up of $\mathbf{P}^2$ at $s_1; s_2$ and s_3 and let $E_1; E_2$ and E_3 be the exceptional divisors on X. The proper transform $\tilde{\Gamma}'$ of Γ' belongs to the linear system $\mathbf{P} = |\pi^*((2k-\mu)L) - \sum_{i=1}^{3} (k-\mu)E_i|$ on X. Let $\tilde{L}_i$ be the proper transform of L_i, then $\tilde{\Gamma}' = \tilde{L}_1 + \ldots + \tilde{L}_{2k-\mu}$. Now we use the terminology from Tannenbaum's paper [17]. In case $2\mu \neq k$, take the nodes on $\tilde{L}_{2k-\mu}$ as the unassigned nodes of $\tilde{\Gamma}'$; in case $2\mu = k$ then take the nodes on $\tilde{L}_{3\mu}$ together with the nodes $\tilde{L}_{2\mu} \cap \tilde{L}_{2\mu+i}$ $(1 \leq i \leq \mu)$ as the unassigned nodes. Then $\tilde{\Gamma}'$ has $[(2k-\mu-1)(2k-\mu-2) - 3(k-\mu)(k-\mu-1)]/2 = p_a(\tilde{\Gamma}')$ assigned nodes. From Corollary 2.14 in Tannenbaum's paper it follows that for each $0 \leq \delta \leq p_a(\tilde{\Gamma}')$ there is a curve $\tilde{\Gamma}$ in $\mathbf{P}$ such that $\tilde{\Gamma}$ is irreducible and nodal with exactly δ nodes. The normalisation C of $\tilde{\Gamma}$ has genus $p_a(\tilde{\Gamma}') - \delta$. The image Γ of $\tilde{\Gamma}$ on $\mathbf{P}^2$ has the degree and singularities we are looking for.

We now prove that for a suited such choice of $\tilde{\Gamma}$ the pencils of lines through s_i induce a g^1_k of type I on C. We denote that linear system by g_i. Suppose for some i we have $\dim(|2g_i|) \geq 3$. From the proof of Lemma 4.2 in [1] it follows that $\dim(|2(g_1 + g_2 + g_3)|) \geq 19$. Assume there exists a value for δ such that for each choice of δ assigned nodes on $\tilde{\Gamma}'$ we find $\dim(|2(g_1 + g_2 + g_3)|) \geq 19$ on C. Choose general divisors $D_{i,1}; D_{i,2}$ in g_i corresponding to lines $L_{i,1}; L_{i,2}$ on $\mathbf{P}^2$ through s_i. Since $g > 6k - 19$, the linear system $|2(g_1 + g_2 + g_3)|$ is special. The space, V say, of canonical adjoint curves for Γ containing the points in $\sum_{i=1}^{3} (D_{i,1} + D_{i,2})$ has affine dimension $19 - 6k + [(2k-\mu-1)(2k-\mu-2) - 3(k-\mu)(k-\mu-1)]/2 - \delta$. Canonical adjoint curves have degree $2k - \mu - 3$ and they have multiplicity at least $k - \mu - 1$ at s_i. Hence, if such a curve γ contains $\sum_{i=1}^{3} (D_{i,1} + D_{i,2})$ then $\gamma = \sum_{i=1}^{3} (L_{i,1} + L_{i,2}) + \gamma$' for some plane curve γ' of degree $2k - \mu - 9$ having multiplicity $k - \mu - 3$ at s_i and containing the δ nodes of Γ. Since $k \geq 6$, for plane curves of degree $2k - \mu - 9$ to have multiplicity $k - \mu - 3$ at each point s_i are $3(k-\mu-2)(k-\mu-3)/2$ independent conditions. Indeed, using $k - \mu - 4$ lines we can obtain all tangent cones of degree less than $k - \mu - 3$ at s_1. Then using $k - \mu - 3$ general lines through s_1 and another $k - \mu - 4$ lines, we can obtain all tangent cones of degree less than $k - \mu - 3$ at s_2. Finally using $k - \mu - 3$ times the lines $s_1 s_2$ and another $k - \mu - 4$ lines we can obtain all tangent cones of degree less than $k - \mu - 3$ at s_3. Smoothing the unassigned nodes on $\tilde{\Gamma}'$ we obtain a rational irreducible curve Γ_0 on $\mathbf{P}^2$. Our assumption would imply that each choice of δ from the nodes of Γ_0 would impose at most $\delta - 1$ conditions on plane curves γ' of degree $2k - \mu - 9$ having multiplicity $k - \mu - 3$ at the points $s_1; s_2$ and s_3. Since it concerns linear conditions, this would imply that all $p_a(\tilde{\Gamma}')$ nodes of Γ_0 impose at most $\delta - 1$ independent conditions on those plane curves γ'. But then there exists such a plane curve γ' containing all those nodes and $\gamma'+\sum_{i=1}^{3}(L_{i,1}+L_{i,2})$ would be a canonically adjoint curve for Γ_0. However, since Γ_0 is a rational curve, this is impossible.

Remark 4.2. Proposition 4.1 suffices for the discussion in Example 3.10 except for the case $k=6$; $g=17$; $\mu=3$ and $k=7$; $g\in\{21;22;23\}$; $\mu=4$. For those cases we use the following arguments suggested (in a different form, so any mistake would be due to the actual author) by T. Kato.

Fix 5 general points P_1; P_2; P_3; Q_1; Q_2 in $\mathbf{P}^2$. Let $L_i(1\leq i\leq 5)$ be the following lines: $L_1=P_1P_2$; $L_2=P_3Q_1$; $L_3=P_3Q_2$; $L_4=P_1Q_1$; $L_5=P_2Q_2$. Let C_4 be a plane curve of degree 4 containing the points P_i. Then $\Gamma_0=(\sum_{i=1}^{5}L_i)+C_4$ is a plane curve of degree 9 such that the points P_i have multiplicity 3 on Γ_0 and the points Q_i have multiplicity 2 on Γ_0. Permuting the points $P_1;P_2;P_3$ and using Bertini's theorem we find that there exist irreducible plane curves Γ having an ordinary singularity of multiplicity 3 at P_i, an ordinary node at Q_i and no other singularities. The geometric genus g of the normalisation C is 17. The canonical adjoint curves γ of Γ containing 2 lines through P_1 are plane curves of degree 4 having a singularity at $P_2;P_3$ and containing $Q_1;Q_2$. Since those points are general on $\mathbf{P}^2$ the space V of such plane curves has dimension 6. This implies that the pencil of lines through P_1 induces a linear system g^1_6 on C such that $\dim(|2g^1_6|)=2$. Hence g^1_6 is of type I on C. The same is true for the pencil of lines through P_2 and through P_3. This proves the existence of a curve as mentioned in Example 3.10 in case $g=17$; $k=6$; $\mu=3$.

Next, take general points $P_1;P_2;P_3;Q_1;\ldots;Q_6$ on $\mathbf{P}^2$. Let C_2 be the conic containing $Q_1;\ldots;Q_5$. Let L_i $(1\leq i\leq 8)$ be the lines $L_1=P_1P_2$; $L_2=P_1Q_1$; $L_3=P_1Q_2$; $L_4=P_2Q_3$; $L_5=P_2Q_6$; $L_6=P_3Q_4$; $L_7=P_3Q_5$; $L_8=P_3Q_6$. Then $\Gamma_0=C_2+(\sum_{i=1}^{8}L_i)$ is a plane curve of degree 10 having a singularity of multiplicity 3 at P_i and a singularity of multiplicity 2 at Q_i. Again permuting the points P_i and permuting the points Q_i and using Bertini's theorem we prove the existence of an irreducible plane curve Γ of degree 10 having ordinary singularities of multiplicity 3 at P_i and of multiplicity 2 at Q_i. The genus of the normalisation C is equal to 21. As before the pencil of lines through a point P_i induces a linear system g^1_7 on C satisfying $\dim(|2g^1_7|)=2$. This suffices for Example 3.10 in case $g=21$; $k=7$ and $\mu=4$. The higher genus cases $g\geq 22$ can be solved using Tannenbaum's theorem as used in the proof of Proposition 4.1.

Notation 4.3. For some $k;g$ assume there exist 2 different values $\mu_1;\mu_2$ both at least $k/2$ and at most $k-2$ such that $g\leq g(k;\mu_i):=[(2k-\mu_i-1)(2k-\mu_i-2)-3(k-\mu_i)(k-\mu_i-1)]/2$. Let $\tilde{Z}_i$ be an irreducible component of the space parametrizing integral plane curves Γ of degree $2k-\mu_i$; and geometric genus g with 3 singular points of multiplicity $k-\mu_i$ not on one line. Assume that a general element Γ of $\tilde{Z}_i$ has 3 ordinary singularities of multiplicity $k-\mu_i$ and only nodes as its other singularities and that the normalization C is k-gonal with exactly three linear systems g^1_k – each one of type I. This gives rise to two irreducible subsets $Z_1; Z_2$ of $M_{g;k}(3)$.

Proposition 4.3. *A general curve in Z_2 is not the specialization of curves in Z_1. In other words $Z_2 \not\subset \bar{Z}_1$ with $\bar{Z}_1$ the closure of Z_1 in M_g.*

Proof. First assume that $\mu_2 < \mu_1$. Let $g_1; g_2$ be two linear systems g^1_k on a curve C' belonging to Z_1. Then there exists $D_i \in g_i$ with $\deg(D_1 \cap D_2) \geq \mu_1$ (use the line connecting the singular points corresponding to g_1 and g_2 on the plane model of C'). Let C be a general point on Z_2 and let Γ be the associated plane model; let $s_1; s_2; s_3$ be the points of multiplicity $k - \mu_2$ on Γ and let $g_1; g_2; g_3$ be the associated linear systems g^1_k on C. Assume C belongs to closure of Z_1. Then for two of the linear systems g^1_k – say g_1 and g_2 – there exist divisors $E_1 \in g_1$ and $E_2 \in g_2$ with $\deg(E_1 \cap E_2) \geq \mu_1$. We are going to prove that this is impossible, hence Z_2 is not contained in the closure of Z_1.

The divisors E_1 and E_2 correspond to lines resp. L_1 and L_2 through resp. s_1 and s_2. In case $l_1 = l_2 = s_1 s_2$ then $\deg(E_1 \cap E_2) = \mu_2 < \mu_1$, this is a contradiction. In case $l_1 = s_1 s_3$ and $l_2 = s_2 s_3$ then $\deg(E_1 \cap E_2) = k - \mu_2 \leq k/2 \leq \mu_2 < \mu_1$, again a contradiction. Therefore $l_1 \cap l_2$ is either not a point of Γ, a smooth point of Γ or a node of Γ. In those cases $\deg(E_1 \cap E_2) \leq 2 < k/2 \leq \mu_1$. Once more a contradiction.

Next, assume $\mu_1 < \mu_2$. Take a general curve C from Z_2, let Γ be the plane model of degree $2k - \mu_2$ and let $s_1; s_2; s_3$ be its singular points of multiplicity $k - \mu_2$. Let g_i be the linear system g^1_k on C obtained from the pencil of lines through s_i. Assume $C = C_0$ in a family $(C_t)_{t \in T}$ for some affine curve T and $C_t \in Z_1$ for $t \neq 0$ with a plane model Γ_t of degree $2k - \mu_1$ with 3 singular points at $s_{1;t}$; $s_{2;t}$; $s_{3;t}$ of multiplicity $k - \mu_1$. Let $g_{i;t}$ be the linear system g^1_k on C_t obtained from the pencil of lines through $s_{i;t}$. Assume g_i is the limit of $g_{i;t}$. On $g_{1;t}$; $g_{2;t}$ we find divisors $D_{1;t}$; $D_{2;t}$ with $\deg(D_{1;t} \cap D_{2;t}) = k - \mu_1$ (obtained from the lines $s_{1;t}s_{3;t}$ and $s_{2;t}s_{3;t}$). Let D_1; D_2 be the limits on C. Since all linear systems g^1_k on C are of type I, they belong to different linear systems g^1_k on C, say g_1 and g_2 (obtained from the pencil of lines through s_1 and s_2). Since $k - \mu_2 < k - \mu_1$, it follows that those divisors D_1 and D_2 are obtained using the line $s_1 s_2$. Next take $D_{3;t} \in g_{3;t}$ with $\deg(D_{1;t} \cap D_{3;t}) = \mu_1$. Let D_3 be the limit of $D_{3;t}$. Again since the linear systems g^1_k on C are of type I one has $D_3 \in g_3$, hence D_3 corresponds to a line through s_3. But $\deg(D_3 \cap D_1) \geq \mu_1$ and $k - \mu_2 < \mu_1$ hence that line contains at least 2 points of $s_1 s_2$. Since s_1; s_2 and s_3 are not on one line, this is impossible.

Discussion 4.5. We now give the argument proving the claims in Example 3.11. There exist plane nodal curves Γ of degree 8 with exactly 4 nodes which are 4 general points in $\mathbf{P}^2$ (see [18] and [2], Theorem 3.2). In particular, the space of plane curves of degree 3 containing 3 of those nodes has dimension 6. In a way explained in the proof of Proposition 4.1 this implies that on the normalization C of Γ and for the linear system g^1_6 defined by the pencil of lines through the fourth node, one has $\dim|K_C - 2g_6^1| = 6$, hence $\dim|2g_6^1| = 12 - 17 + 1 + 6 = 2$. This proves that each g_6^1 on C is of type I. Suppose that $C = C_0$ in a family of smooth curves $\mathcal{C} \to T$ with C_t for $t \neq 0$ a 6-gonal curve with exactly 3 linear systems g_6^1. For $t \neq 0$ the curve C_t is the normalization of a plane curve Γ_t of degree 9 having 3 points $s_{1;t}$; $s_{2;t}$; $s_{3;t}$ of multiplicity 3. Let $g_{1;t}$; $g_{2;t}$; $g_{3;t}$ be the linear systems g_6^1 on C_t defined by the pencil of lines through $s_{1;t}$; $s_{2;t}$; $s_{3;t}$; let g_1; g_2; g_3 be their limit on C; let s_1; s_2; s_3 be the nodes on Γ defining g_1; g_2; g_3. The lines $s_{1;t}s_{3;t}$ and $s_{2;t}s_{3;t}$ define divisors $E_{1;t} \in g_{1;t}; E_{2;t} \in g_{2;t}$ with $\deg(E_{1;t} \cap E_{2;t}) = 3$. Let $E_1 \in g_1; E_2 \in g_2$ be the limits, then $\deg(E_1 \cap E_2) \geq 3$. It follows that both $E_1; E_2$ are defined by the line s_1s_2. The line $s_{1;t}s_{3;t}$ also defines $E_{3;t} \in g_{3;t}$ with $\deg(E_{1;t} \cap E_{3;t}) = 3$. Let $E_3 \in g_3$ be the limit, then $\deg(E_1 \cap E_3) \geq 3$. But E_3 is defined by a line through s_3 and this would imply $s_3 \in s_1s_2$, a contradiction.

Discussion 4.6. In order to find proofs for the claims made in Examples 3.12, 3.13 and 3.14 we prove the following statement.

Fix integers g; d; m and x. We prove the existence of an irreducible plane curve Γ such that Γ has degree d; the normalisation C of Γ has genus g; Γ has exactly m ordinary singular points of multiplicity x and ordinary nodes as its other singularities; the pencil of lines through a singular point of multiplicity x induces a g_{d-x}^1 on C of type I in the following cases:

(a) $21 \leq g \leq 24$; $d = 10$; $m = 4$; $x = 3$,
(b) $g = 21$; $d = 10$; $m = 5$; $x = 3$,
(c) $40 \leq g \leq 42$; $d = 13$; $m = 4$; $x = 4$,
(d) $40 \leq g \leq 43$; $d = 12$; $m = 4$; $x = 3$,
(e) $g = 40$; $d = 12$; $m = 4$; $x = 3$,
(f) $65 \leq g \leq 67$; $d = 15$; $m = 4$; $x = 4$,
(g) $65 \leq g \leq 66$; $d = 14$; $m = 4$; $x = 3$,
(h) $g = 65$; $d = 16$; $m = 4$; $x = 5$.

The proofs and arguments are identical to the previous ones, the reader can verify all the claims in Examples 3.12; 3.13 and 3.14.

In order to prove (a), choose an irreducible nodal curve Γ_0 of degree 6 having exactly $x \in \{4;\ 5;\ 6;\ 7\}$ nodes at general points in $\mathbf{P}^2$ (see [2]; [18]). Let $s_1; s_2; s_3$ and s_4 be nodes of Γ_0. Take a general line L_i through s_i. Let X be the blowing-up of $\mathbf{P}^2$ at $s_1; s_2; s_3$ and s_4 and let $\tilde{\Gamma}_0$; $\tilde{L}_i$ be the proper transforms of Γ_0; L_i. Then $\tilde{\Gamma}_0 + \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 \in \mathbf{P} := |10L - 3E_1 - 3E_2 - 3E_3 - 3E_4|$ (here L is the inverse image of a line in $\mathbf{P}^2$; E_i are the exceptional divisors). Considering the nodes of $\tilde{\Gamma}_0$ as the only assigned nodes, it follows from [13] that $\mathbf{P}$ contains an irreducible curve with exactly x nodes degenerating to $\tilde{\Gamma}_0$ such that the limits of the nodes are the assigned nodes of $\tilde{\Gamma}_0$. The image on $\mathbf{P}^2$ is a plane curve Γ as we searched for. The claim about the linear systems g_7^1 holds because Γ has its singular points at general points (compare with 4.2). For the other cases you start with a suited nodal curve with nodes in general position and you add lines.

This argument is not possible any more to prove case (h). To prove case (h) you start with an irreducible nodal curve Γ_0 of degree 10 having exactly 3 nodes and 3 general conics through those nodes.

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